Some aspects of the stochastic maximum principle for SDEs of mean-field type and application to portfolio choice

Boualem Djehiche
KTH Royal Institute of Technology
Stockholm

October 6, 2013

Outline

I. The pre-commitment case

II. The non-commitment case

Part I is based on

- Andersson, D. and BD (2010): A maximum principle for SDEs of mean-field type. Applied Math. and Optimization.
- Buckdahn, R., BD, Li, J. and Peng, S. (2011): A general stochastic maximum principle for SDEs of mean-field type. Applied Math. and Optimization

Part II is based on

- BD and Huang M. (2013): A characterization of sub-game perfect Nash equilibria for SDEs of mean-field type (submitted).

Competing papers:

- Hu, Y., Jin, H., and Zhou, X. Y. (2011): *Time-inconsistent linear-quadratic control.* Preprint (to appear in SICON).
- Yong, J. (2011): *A linear-quadratic optimal control problem for mean-field stochastic differential equations.* Preprint.
- A. Bensoussan, A., Sung, K. C. J and Yam, S. C. P. (2012): *Linear-Quadratic Time-Inconsistent Mean Field Games.* Preprint.

Part I. The pre-commitment case

The dynamics of the controlled SDE of mean-field type (through the expected value) on $\mathbb{R}$ is

$$\begin{cases} dX(t) = b(t, X(t), \mathbb{E}[X(t)], u(t))dt + \sigma(t, X(t), \mathbb{E}[X(t)], u(t))dB(t). \\ X(0) = x_0, \end{cases} \tag{1}$$

The cost functional is also of mean-field type:

$$J(u) = \mathbb{E}\left[\int_0^T h\left(t, X(t), \mathbb{E}[X(t)], u(t)\right) dt + g\left(X(T), \mathbb{E}[X(T)]\right)\right]. \tag{2}$$

We want to "find" or characterize (through a Maximum Principle)

$$u^* = \arg\min_{u\in\mathcal{U}} J(u). \tag{3}$$

where, $\mathcal{U}$ is the class of "admissible controls": measurable, adapted processes $u : [0, T] \times \Omega \longrightarrow U$ (non-convex in general) satisfying some integrability conditions.

Time inconsistent control problem

Whenever, the classical **Bellman optimality principle** (or the Dynamic Programming Principle) holds, the control problem is **time consistent**.

The classical Bellman optimality principle for Markov dynamics of the form:

$$dX_t = b(t, X(t), u(t))dt + \sigma(b(t, X(t), u(t)))dB(t).$$

b and σ are nonrandom. Consider the cost functional

$$J(u) = \mathbb{E}\left[\int_0^T h\left(t, X(t), u(t)\right) dt + g\left(X(T)\right)\right].$$

The value-function of the problem (which solves the HJB equation) is

$$\begin{aligned} V(x,t) &:= \min_u \mathbb{E}\left[\int_t^T h\left(r, X(r), u(r)\right) dr + g\left(X(T)\right) \middle| X(t) = x\right] \\ &= \min_u \mathbb{E}\left[\int_t^T h\left(r, X(r), u(r)\right) dr + g\left(X(T)\right) \middle| \mathcal{F}_t\right] \end{aligned}$$

by the Markov property.

The Bellman Principle states that for any $0 \leq t \leq t+s \leq T$,

$$V(x,t) = \min_u \mathbb{E}\left[\int_t^{t+s} h\left(r, X(r), u(r)\right) dr + V\left(X(t+s), t+s\right)\right) |X_t = x\right].$$

This principle is based on the "law of iterated conditional expectations": Assume $\bar{u}$ is an optimal control for our problem. Then, denoting by $\bar{X}$ the corresponding optimal dynamics, we have

$$\begin{aligned}
V(x,t) = J(t,x,\bar{u}) &= \mathbb{E}\left[\int_t^T h\left(r, \bar{X}(r), \bar{u}(r)\right) dr + g\left(\bar{X}(T)\right) |\bar{X}(t) = x\right] \\
&= \mathbb{E}\left[\int_t^T h\left(r, \bar{X}(r), \bar{u}(r)\right) dr + g\left(\bar{X}(T)\right) |\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{t+s} h\left(r, \bar{X}(r), \bar{u}(r)\right) dr + \int_{t+s}^T h\left(r, \bar{X}(r), \bar{u}(r)\right) dg\left(\bar{X}(T)\right) |\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^{t+s} h\left(r, \bar{X}(r), \bar{u}(r)\right) dr + \mathbb{E}\left[\int_{t+s}^T h\left(r, \bar{X}(r), \bar{u}(r)\right) dg\left(\bar{X}(T)\right) |\mathcal{F}_{t+s}\right] |\mathcal{F}_t
\end{aligned}$$

(apply the law of iterated conditional expectations)

$$= \mathbb{E}\left[\int_t^{t+s} h\left(r, \bar{X}(r), \bar{u}(r)\right) dr + V\left(\bar{X}(t+s), t+s\right) |\mathcal{F}_t\right].$$

Two (among many other) cases where the Bellman principle does not hold

- The terminal cost is of the form $g\left(X(T), \mathbb{E}\left(X_T\right)\right)$ i.e. it is **nonlinear in** $\mathbb{E}\left(X_T\right)$.
- The hyperbolic discounting case:

$$V(x,t) = \min_u \mathbb{E}\left[\int_t^T \varphi(r-t)h\left(r, X(r), u(r)\right) dr + \varphi(T-t)g\left(X(T)\right) | X(t) = x\right]$$

for a positive function φ. It can be shown in fact that

$$u^*(t,x,\cdot) \neq \arg\min_u J(t',x',u)$$

although the terminal cost g is standard.

A classical example: Mean-variance portfolio selection

The dynamics of the self-financing portfolio is

$$\begin{cases} dX(t) = (rX(t) + (\alpha - r)\, u(t))\, dt + \sigma u(t) dB(t), \\ X(0) = x_0. \end{cases} \tag{4}$$

All the coefficients are constant.

The control $u(t)$ denotes the amount of money invested in the risky asset at time t.

The cost functional, to be minimized, is given by

$$J(u) = \frac{\gamma}{2}\mathrm{Var}(X(T)) - \mathbb{E}[X(T)]. \tag{5}$$

By rewriting it as

$$J(u) = \mathbb{E}\left(\frac{\gamma}{2}X^2(T) - X(T)\right) - \frac{\gamma}{2}\left(\mathbb{E}[X(T)]\right)^2,$$

it becomes nonlinear in $\mathbb{E}[X(T)]$.

The mean-field SDE is obtained as an L^2-limit of an interacting particle system of the form

$$dx^{i,n}(t) = b^{i,n}(t,\omega,u(t))dt + \sigma^{i,n}(t,\omega,u(t))dB_t^i,$$

when $n \to \infty$, where, the B^i's are independent Brownian motions, and

$$b^{i,n}(t,\omega,u(t)) := b\left(t, x^{i,n}(t), \frac{1}{n}\sum_{j=1}^n x^{j,n}(t), u(t)\right)$$
$$\sigma^{i,n}(t,\omega,u(t)) := \sigma\left(t, x^{i,n}(t), \frac{1}{n}\sum_{j=1}^n x^{j,n}(t), u(t)\right).$$

The classical example is the McKean-Vlasov model, in which the coefficients are linear in the law of the process. (see e.g. Sznitman (1989) and the references therein).

For the nonlinear case, see Jourdain, Mèlèard and Woyczynski (2008).

Extending the HJB equation to the mean-field case

(1) Ahmed and Ding (2001) express the value function in terms of the Nisio semigroup of operators and derive a (very complicated) HJB equation.

(2) Huang *et al.* (2006) use the Nash Certainty Equivalence Principle to solve an extended HJB equation.

(3) Lasry and Lions (2007) suggest a new class of nonlinear HJB involving the dynamics of the probability laws $(\mu_t)_t$.

(4) Björk and Murgoci (2008), Björk, Murgoci and Zhou(2011) use the notion of Nash equilibrium to transform the time inconsistent control problem into a standard one and derive an "extended" HJB equation.

Assumptions

- The action space U is a subset of $\mathbb{R}$ (not necessarily **convex**!).
- All the involved functions are sufficiently smooth:
  - b, σ, g, h are twice continuously differentiable with respect to (x, y).
  - b, σ, g, h and all their derivatives with respect to (x, y) are continuous in (x, y, v), and bounded.

We let $\hat{u}$ denote an optimal control, and $\hat{x}$ the corresponding state process. Also, denote

$$\delta\varphi(t) = \varphi(t, \hat{x}(t), E[\hat{x}(t)], u(t)) - \varphi(t, \hat{x}(t), E[\hat{x}(t)], \hat{u}(t)),$$

$$\varphi_x(t) = \frac{\partial\varphi}{\partial x}(t, \hat{x}(t), \mathbb{E}[\hat{x}(t)], \hat{u}(t)),$$

$$\varphi_y(t) = \frac{\partial\varphi}{\partial y}(t, \hat{x}(t), \mathbb{E}[\hat{x}(t)], \hat{u}(t)),$$

and similarly for higher derivatives.

Associated Hamiltonian

The Hamiltonian associated with the r.v. X:

$$H(t,X,u,p,q) := b(t,X,\mathbb{E}[X],u)p + \sigma(t,X,\mathbb{E}[X],u)q + h(t,X,\mathbb{E}[X],u); \quad (6)$$

Denote

$$\begin{aligned} &\delta H(t) := p(t)\delta b(t) + q(t)\delta\sigma(t) + \delta h(t), \\ &H_j(t) = b_j(t)p + \sigma_j(t)q + h_j(t), \quad j = x, y, \\ &H_{xx}(t) = b_{xx}(t)p + \sigma_{xx}(t)q + h_{xx}(t). \end{aligned} \quad (7)$$

Adjoint equations

(a) The first-order adjoint equation is of mean-field type:

$$\begin{cases} dp(t) = & -\{H_x(t) + \mathbb{E}[H_y(t)]\}\,dt + q(t)dB(t) \\ p(T) = & g_x(T) + \mathbb{E}[g_y(T)]. \end{cases} \tag{8}$$

Under our assumptions, this is a linear mean-field backward SDE with bounded coefficients. It has a unique adapted solution such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |p(t)|^2 + \int_0^T |q(t)|^2\,dt\right] < +\infty. \tag{9}$$

(See Buckdahn *et al.* (2009), Theorem 3.1)

(b) The second-order adjoint equation is "standard":

$$\begin{cases} dP(t) = -\left(2b_x(t)P(t) + \sigma_x^2(t)P(t) + 2\sigma_x(t)Q(t) + H_{xx}(t)\right) dt \\ \qquad\qquad +Q(t)\, dB(t), \\ \\ P(T) = g_{xx}(T). \end{cases} \tag{10}$$

This is a standard linear backward SDE, whose unique adapted solution (P, Q) satisfies the following estimate

$$E\left[\sup_{t\in[0,T]} |P(t)|^2 + \int_0^T |Q(t)|^2\, dt\right] < \infty. \tag{11}$$

Necessary Conditions for Optimality

Theorem. Let the above assumption hold. If $(\hat{X}(\cdot), \hat{u}(\cdot))$ is an optimal solution of the control problem (1)-(3), then there are pairs of $\mathcal{F}$-adapted processes $(p(\cdot), q(\cdot))$ and $(P(\cdot), Q(\cdot))$ that satisfy (8)-(9) and (10)-(11), respectively, such that

$$
\begin{aligned}
& H(t, \hat{X}(t), u, p(t), q(t)) - H(t, \hat{X}(t), \hat{u}(t), p(t), q(t)) \\
& \quad + \tfrac{1}{2} P(t) \left(\sigma(t, \hat{X}(t), \mathbb{E}[\hat{X}(t)], u) - \sigma(t, \hat{X}(t), \mathbb{E}[\hat{X}(t)], \hat{u}(t)) \right)^2 \le 0, \\
& \quad \forall u \in U, \quad a.e.\ t \in [0, T], \quad \mathbb{P} - a.s.
\end{aligned}
\tag{12}
$$

Sufficient Conditions for Optimality

Assuming convexity of the action space U and the coefficients, Condition (12) is also sufficient (without the third term).

In this case Condition (12) is equivalent to

$$\partial_u H(t, \hat{X}(t), \hat{u}(t), p(t), q(t)) = 0, \qquad 0 \leq t \leq T. \tag{13}$$

We don't need the solutions (P, Q) of the second-order adjoint equation (10)-(11).

A worked out example- Mean-variance portfolio selection

The state process equation is

$$dX(t) = (rX(t) + (\alpha - r)\, u(t))\, dt + \sigma u(t) dB(t), \;\; X(0) = x_0. \tag{14}$$

The cost functional, to be minimized, is given by

$$J(u) = \mathbb{E}\left(\frac{\gamma}{2} X(T)^2 - X(T)\right) - \frac{\gamma}{2}\left(\mathbb{E}[X(T)]\right)^2,$$

The Hamiltonian for this system is

$$H(t, x, u, p, q) = (rx + (\alpha - r)\, u)\, p + \sigma u q.$$

The adjoint equation becomes

$$\begin{cases} dp(t) &= -rp(t)dt + q(t)dB_t, \\ p(T) &= \gamma(X(T) - \mathbb{E}[X(T)]) - 1, \end{cases}$$

Try a solution of the form

$$p_t = A_t(X(t) - \mathbb{E}[X(t)]) - C_t,$$

with A_t, C_t deterministic functions such that

$$A_T = \gamma, \quad C_T = 1.$$

After easy manipulations, together with the first order condition for minimizing the Hamiltonian yielding

$$(\alpha - r)\,p(t) + \sigma q(t) = 0, \tag{15}$$

and

$$q(t) = A_t \sigma u(t), \tag{16}$$

we get

$$\begin{cases} (r-\alpha)^2 A_t - \left(2rA_t + \dot{A}_t\right)\sigma^2 = 0, & A_T = \gamma, \\ rC_t + \dot{C}_t = 0, & C_T = 1. \end{cases}$$

The solutions to these equations are

$$\begin{cases} A_t = \gamma e^{(2r-\Lambda)(T-t)}, \\ C_t = e^{\rho(T-t)}, \end{cases} \tag{17}$$

where,

$$\Lambda = \frac{(\alpha - r)^2}{\sigma^2} = (\text{Sharpe ratio})^2 .$$

The optimal control becomes

$$\hat{u}(t,\hat{X}(t)) = \frac{\alpha - r}{\sigma^2}\left(x_0 e^{r(T-t)} + \frac{1}{\gamma}e^{(\Lambda - r)(T-t)} - \hat{X}(t)\right), \tag{18}$$

which is identical to the optimal control found in Zhou and Li (2000), obtained by embedding the problem into a stochastic LQ problem.

Outline of the proof of the Maximum Principle

Following Peng (1990), the variational inequality (12) is derived in several steps from the fact that

$$J(u^{\epsilon}(\cdot)) - J(\hat{u}(\cdot)) \geq 0,$$

where, $u^{\epsilon}(\cdot)$ is the so-called spike variation of $\hat{u}(\cdot)$, defined as follows.

For $\epsilon > 0$, pick a subset $E_{\epsilon} \subset [0, T]$ such that $|E_{\epsilon}| = \epsilon$ and consider the control process (spike variation of u)

$$u^{\epsilon}(t) := \begin{cases} u(t), & t \in E_{\epsilon}, \\ \hat{u}(t), & t \in E_{\epsilon}^{c}, \end{cases}$$

where, $u(\cdot) \in \mathcal{U}$ is an arbitrary admissible control.

Denote $x^{\epsilon}(\cdot) := x^{u^{\epsilon}}(\cdot)$ the corresponding state process which satisfies (1).

The key relation between performance functional J and the Hamiltonian H is

$$J(u^{\epsilon}) - J(\hat{u}) = -\mathbb{E}\left[\int_0^T \left(\delta H(t) + \frac{1}{2}P(t)(\delta\sigma(t))^2\right) 1\!\!1_{E_\epsilon}(t)\, dt\right] + R(\epsilon), \quad (19)$$

where,

$$|R(\epsilon)| \leq \epsilon\bar{\rho}(\epsilon),$$

for some function $\bar{\rho} : (0, \infty) \to (0, \infty)$ such that $\bar{\rho}(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$.

This is a **finer estimate** than the standard one related to the original Peng's Stochastic Maximum Principle.

Part II. The non-commitment case

The dynamics of the controlled SDE is

$$dX^u(s) = b(s,\omega,u)ds + \sigma(s,\omega,u)dW(s). \tag{20}$$

where,

$$\begin{cases} b(s,u) := b(s,X(s),\mathbb{E}[X(s)],u(s)) \\ \sigma(s,u) := \sigma(s,X(s),\mathbb{E}[X(s)],u(s)). \end{cases}$$

The cost functional is

$$J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T h\left(s,X^u(s),\mathbb{E}_{t,x}[X^u(s)],u(s)\right)ds + g\left(X^u(T),\mathbb{E}_{t,x}[X^u(T)]\right)\right], \tag{21}$$

We want to "find" or characterize (through a Maximum Principle) the t-optimal policy

$$u^*(t,x,\cdot) := \arg\min_{u\in\mathcal{U}} J(t,x,u). \tag{22}$$

Failure to remain optimal across time!

A key-observation made by Ekeland, Lazrak and Pirvu (2007)-(2008) is that:

Time inconsistent optimal solutions (although they exist mathematically) are irrelevant in practice, since the t-optimal policy may not be optimal after t:

$$u^*(t, x, \cdot) \neq \arg\min_u J(t', x', u)$$

The decision-maker would not implement the t-optimal policy at a later time, if he/she is not forced to do so.

Game theoretic approach

Following Ekeland, Lazrak and Pirvu (2007)-(2008), and Björk and Murgoci (2008), we may view the problem as a game and look for a **subgame perfect Nash equilibrium point** $\hat{u}$ in the following sense:

- Assume that all players (selves) s, such that $s > t$, use the control $\hat{u}(s)$.
- Then it is optimal for player (self) t to also use $\hat{u}(t)$.

Subgame perfect Nash equilibrium

A subgame perfect Nash equilibrium is an equilibrium such that players strategies constitute a Nash equilibrium in every subgame of the original game. It may be found by backward induction for sequential games. First, one determines the optimal strategy of the player who makes the last move of the game. Then, the optimal action of the next-to-last moving player is determined taking the last player's action as given. The process continues in this way backwards in time until all players actions have been determined.

To characterize the equilibrium point $\hat{u}$, Ekeland *et al.* suggest the following definition that uses a "local" spike variation in a natural way:

- Fix (t, x) and define the control law u^{ϵ} as the "local" spike variation of $\hat{u}$ over the set $E_{t,\epsilon} := [t, t+\epsilon]$, (note that $|E_{t,\epsilon}| = \epsilon$),

$$u^{\epsilon}(s) := \begin{cases} u(s), & s \in E_{t,\epsilon}, \\ \\ \hat{u}(s), & s \in [t, T] \setminus E_{t,\epsilon}, \end{cases}$$

  where, $u(\cdot) \in \mathcal{U}$ is an arbitrary admissible control (or simply any real number).

  Definition. The control law $\hat{u}$ is an **equilibrium point** if

$$\lim_{\epsilon \downarrow 0} \frac{J(t, x, \hat{u}) - J(t, x, u^{\epsilon})}{\epsilon} \geq 0, \tag{23}$$

  for all choices of t, x and u.

We let $\hat{u}$ denote an equilibrium point, and $\hat{x}$ the corresponding state process.

For $\varphi = h, g$, we define

$$\begin{cases} \delta\varphi^{t,x}(s) = \varphi(s, \hat{X}(s), E_{t,x}[\hat{X}(s)], u(s)) - \varphi(s, \hat{X}(s), E_{t,x}[\hat{X}(s)], \hat{u}(s)), \\ \varphi_y^{t,x}(s) = \frac{\partial\varphi}{\partial y}(s, \hat{X}(s), E_{t,x}[\hat{X}(s)], \hat{u}(s)), \quad \varphi_{yy}^{t,x}(s) = \frac{\partial^2\varphi}{\partial y^2}(s, \hat{X}(s), E_{t,x}[\hat{X}(s)], \hat{u}(s)), \\ \varphi_z^{t,x}(s) = \frac{\partial\varphi}{\partial z}(s, \hat{X}(s), E_{t,x}[\hat{X}(s)], \hat{u}(s)), \quad \varphi_{zz}^{t,x}(s) = \frac{\partial^2\varphi}{\partial z^2}(s, \hat{X}(s), E_{t,x}[\hat{X}(s)], \hat{u}(s)). \end{cases}$$

The Hamiltonian associated with the r.v. X:

$$H^{t,x}(s, X, u, p, q) := b(s, X, E[X], u)p + \sigma(s, X, E[X], u)q + h(s, X, E_{t,x}[X], u).$$

Also denote

$$\delta H^{t,x}(s) := p\delta b(s) + q\delta\sigma(s) + \delta h^{t,x}(s). \tag{24}$$

The key relation between the performance functional J and the Hamiltonian H associated with (20):

$$J(t,x,\hat{u}) - J(t,x,u^{\epsilon}) = -E_{t,x}\left[\int_t^{t+\epsilon} \delta H^{t,x}(s) + \frac{1}{2}P^{t,x}(s)(\delta\sigma(s))^2\, ds\right] + R^{t,x}(\epsilon),$$

where,

$$|R^{t,x}(\epsilon)| \leq \epsilon\bar{\rho}(\epsilon),$$

for some function $\bar{\rho} : (0,\infty) \to (0,\infty)$ such that $\bar{\rho}(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$.

The processes $(p(t,s), q(t,s))$, $s \in [t, T]$, satisfy the first order adjoint equation:

$$\begin{cases} dp^{t,x}(s) = -\{H_y^{t,x}(s) + E_{t,x}\left[H_z^{t,x}(s)\right]\}ds + q^{t,x}(s)dW_s, \\ p^{t,x}(T) = g_y^{t,x}(T) + E_{t,x}[g_z^{t,x}(T)], \end{cases} \tag{25}$$

where,

$$H_j^{t,x}(s) = b_j(s)p^{t,x}(s) + \sigma_j(s)q^{t,x}(s) + h_j^{t,x}(s), \qquad j = y, z, \tag{26}$$

which has a unique adapted solution such that

$$E_{t,x}\Big[\sup_{s\in[t,T]} |p^{t,x}(s)|^2 + \int_t^T |q^{t,x}(s)|^2\, ds\Big] < \infty. \tag{27}$$

The processes $(P(t,s), Q(t,s))$, $s \in [t, T]$, satisfy the "standard" second-order adjoint equation:

$$\begin{cases} dP^{t,x}(s) = -2b_y(s)P^{t,x}(s) + \sigma_y^2(s)P^{t,x}(s) + 2\sigma_y(s)Q^{t,x}(s) \\ \qquad\qquad +H_{yy}^{t,x}(s)\,\mathrm{ds}+\mathrm{Q}^{t,x}(s)\,dW_s, \\ \\ P^{t,x}(T) = g_{yy}^{t,x}(T). \end{cases} \tag{28}$$

where,

$$H_{yy}^{t,x}(s) = b_{yy}(s)p + \sigma_{yy}(s)q + h_{yy}^{t,x}(s). \tag{29}$$

which has a unique adapted solution such that

$$E_{t,x}\Big[\sup_{s\in[t,T]} |P^{t,x}(s)|^2 + \int_t^T |Q^{t,x}(s)|^2\,ds\Big] < \infty. \tag{30}$$

Characterization of the equilibrium points (I)

Theorem. Assume the above assumption hold.

Then $\hat{u}(\cdot)$ is an equilibrium point for the system (20)-(21) if and only if there are pairs of $\mathcal{F}$-adapted processes (p, q) and (P, Q) which satisfy (25)-(27) and (28)-(30), respectively, and for which

$$
\begin{aligned}
&H^{t,\hat{X}(t)}(t, \hat{X}(t), v, p^{t,\hat{X}(t)}(t), q^{t,\hat{X}(t)}(t)) - H^{t,\hat{X}(t)}(t, \hat{X}(t), \hat{u}(t), p^{t,\hat{X}(t)}(t), q^{t,\hat{X}(t)}(t)) \\
&\quad + \frac{1}{2} P^{t,\hat{X}(t)}(t) \left(\sigma(t, \hat{X}(t), E[\hat{X}(t)], v) - \sigma(t, \hat{X}(t), E[\hat{X}(t)], \hat{u}(t)) \right)^2 \leq 0, \\
&\forall v \in U,\ \forall t \in [0, T],\ \mathbb{P}-a.s.
\end{aligned}
\tag{31}
$$

Characterization of the equilibrium points (II)

Assume that the action space U is a convex subset of $\mathbb{R}$, and the coefficients b, σ, h and g are locally Lipschitz in v. Then, the admissible strategy $\hat{u}(\cdot)$ is an equilibrium point for the system (20)-(21) if and only if there is a pair of $\mathcal{F}$-adapted processes (p, q) that satisfies (25)-(27) and for which

$$H^{t,\hat{X}(t)}(t, \hat{X}(t), \hat{u}(t), p^{t,\hat{X}(t)}(t), q^{t,\hat{X}(t)}(t)) = \max_{v \in U} H^{t,\hat{X}(t)}(t, \hat{X}(t), v, p^{t,\hat{X}(t)}(t), q^{t,\hat{X}(t)}(t)) \tag{32}$$

for all $t \in [0, T]$, $\mathbb{P}-a.s.$

We don't need the solution (P, Q) of the second-order adjoint equation (28)-(30).

A worked out example- Mean-variance portfolio selection

The state process equation is

$$dX(s) = (rX(s) + (\alpha - r)\, u(s))\, ds + \sigma u(s) dW(s), \qquad (33)$$

The performance functional is given by

$$J(t, x, u) = \mathbb{E}_{t,x}[X(T)] - \tfrac{\gamma}{2} Var_{t,x}(X(T))$$

$$= \mathbb{E}_{t,x}\left(X(T) - \tfrac{\gamma}{2}(X(T))^2\right) + \tfrac{\gamma}{2}\left(\mathbb{E}_{t,x}[X(T)]\right)^2,$$

where the constant γ, assumed positive, is the risk aversion coefficient.

$$H^{t,x}(s, X(s), u, p, q) := (rX(s) + (\alpha - r)\, u)\, p + \sigma uq.$$

Hence,

$$H^{t,x}(t, x, u, p, q) = (rx + (\alpha - r)\, u)\, p + \sigma uq.$$

The adjoint equation becomes

$$\begin{cases} dp^{t,x}(s) = -rp^{t,x}(s)ds + q^{t,x}(s)dW(s), \\ p^{t,x}(T) = 1 - \gamma \left(\hat{X}(T) - E_{t,x}[\hat{X}(T)] \right). \end{cases} \tag{34}$$

We try a solution of the form

$$p^{t,x}(s) = C_s - A_s \left(\hat{X}(s) - \mathbb{E}_{t,x}[\hat{X}(s)] \right), \tag{35}$$

where, A_s and C_s are deterministic functions such that

$$A_T = \gamma, \quad C_T = 1.$$

Identifying the coefficients in (33) and (34), we get, for $s \geq t$,

$$(2rA_s + \dot{A}_s) \left(\hat{X}(s) - \mathbb{E}_{t,x}[\hat{X}(s)] \right) + (\alpha - r)A_s(\hat{u}(s) - \mathbb{E}_{t,x}[\hat{u}(s)]) = \dot{C}_s + rC_s,$$

$$q^{t,x}(s) = -A_s \sigma \hat{u}(s). \tag{36}$$

The maximizer of the Hamiltonian solves

$$\partial_u H^{t,x}(t, x, \hat{u}(t), p^{t,x}(t), q^{t,x}(t)) = 0,$$

yielding

$$(\alpha - r)p^{t,x}(t) + \sigma q^{t,x}(t) = 0. \tag{37}$$

From (35), we have

$$p^{t,x}(t) = C_t, \tag{38}$$

which is deterministic.

Hence, from (37) we get

$$q^{t,x}(t) = -\frac{\alpha - r}{\sigma} C_t. \tag{39}$$

In view of (36), the equilibrium point is the deterministic function

$$\hat{u}(t) = \frac{\alpha - r}{\sigma^2} \frac{C_t}{A_t}, \quad 0 \le t \le T. \tag{40}$$

Therefore, (35) reduces to

$$(\dot{A}_s + 2rA_s)(\hat{X}(s) - \mathbb{E}_{t,x}[\hat{X}(s)]) = \dot{C}_s + rC_s, \tag{41}$$

giving the equations satisfied by A_s and C_s

$$\begin{cases} \dot{A}_s + 2rA_s = 0, & A_T = \gamma, \\ \dot{C}_s + rC_s = 0, & C_T = 1. \end{cases}$$

The solutions of these equations are

$$A_t = \gamma e^{2r(T-t)}, \quad C_t = e^{r(T-t)}, \quad 0 \le t \le T. \tag{42}$$

Whence, we obtain the following explicit form of the equilibrium point.

$$\hat{u}(t) = \frac{1}{\gamma}\frac{\alpha - r}{\sigma^2}e^{-r(T-t)}, \qquad 0 \leq t \leq T, \tag{43}$$

which is identical to the one obtained in Björk and Murgoci (2008) by solving an extended HJB equation.

Mean-variance portfolio selection with state dependent risk aversion

Modify the performance functional to become

$$J(t,x,u) = \mathbb{E}_{t,x}[X(T)] - \frac{\gamma(x)}{2} Var_{t,x}(X(T))$$

where, the risk aversion coefficient $\gamma(x) > 0$ is made dependent on the current wealth x.

An economically sound form is

$$\gamma(x) = \gamma/x, \qquad x > 0.$$

Since $\gamma(x) > 0$, the equilibrium points of J are the same as the ones of the the performance functional

$$\mathcal{L}(t,x,u) = \gamma^{-1}(x)\mathbb{E}_{t,x}[X(T)] - \frac{1}{2}Var_{t,x}(X(T)). \qquad (44)$$

The associated Hamiltonian is

$$H^{t,x}(s, X(s), u, p, q) := (rX(s) + (\alpha - r)\,u)\,p + \sigma uq.$$

Hence,

$$H^{t,x}(t, x, u, p, q) = (rx + (\alpha - r)\,u)\,p + \sigma uq.$$

The adjoint equation becomes

$$\begin{cases} dp^{t,x}(s) = -rp^{t,x}(s)ds + q^{t,x}(s)dW(s), \\ \\ p^{t,x}(T) = \gamma^{-1}(x) - \left(\hat{X}(T) - \mathbb{E}_{t,x}[\hat{X}(T)]\right), \end{cases} \tag{45}$$

We try a solution of the form

$$p^{t,x}(s) = C_s\gamma^{-1}(x) - A_s\left(\hat{X}(s) - \mathbb{E}_{t,x}[\hat{X}(s)]\right), \tag{46}$$

where, A_s, B_s and C_s are deterministic functions such that

$$A_T = C_T = 1.$$

Identifying the coefficients in the dynamics of X in (33) and (45), we get, for $s \geq t$,

$$\begin{aligned} (\dot{A}_s + 2rA_s)\left(\hat{X}(s) - \mathbb{E}_{t,x}[\hat{X}(s)]\right) + (\alpha - r)A_s\left(\hat{u}(s) - \mathbb{E}_{t,x}[\hat{u}(s)]\right) \\ = (\dot{C}_s + rC_s)\gamma^{-1}(x), \end{aligned} \tag{47}$$

$$q^{t,x}(s) = -A_s\sigma\hat{u}(s). \tag{48}$$

The maximizer of the Hamiltonian solves

$$\partial_u H(t, t, x, \hat{u}(t), p(t, t), q(t, t)) = 0,$$

yielding

$$(\alpha - r)p^{t,x}(t) + \sigma q^{t,x}(t) = 0, \tag{49}$$

But, from (46), we have

$$p^{t,x}(t) = C_t \gamma^{-1}(x). \tag{50}$$

Therefore,

$$q^{t,x}(t) = -\frac{\alpha - r}{\sigma} C_t \gamma^{-1}(x). \tag{51}$$

In view of (48), the value of the equilibrium point is the function

$$\hat{u}(t) = \frac{\alpha - r}{\gamma\sigma^2}\frac{C_t}{A_t}x, \qquad 0 \leq t \leq T. \tag{52}$$

Hence, (47) reduces to

$$(\dot{A}_s + 2rA_s + \tfrac{(\alpha-r)^2}{\gamma\sigma^2}C_s)\,(X(s) - \mathbb{E}_{t,x}[X(s)]) - (\dot{C}_s + rC_s)\tfrac{x}{\gamma} = 0.$$

The functions A_s, B_s and C_s solve the following system of equations:

$$\begin{cases} \dot{A}_s + 2rA_s + \frac{(\alpha-r)^2}{\gamma\sigma^2}C_s = 0, \\ \dot{C}_s + rC_s = 0, \\ A_T = C_T = 1. \end{cases}$$

This system admits the following explicit solution.

$$\mathrm{A}_t = e^{2r(T-t)} + \frac{(\alpha-r)^2}{r\gamma\sigma^2}\left(e^{2r(T-t)} - e^{r(T-t)}\right), \quad C_t = e^{r(T-t)}, \quad 0 \leq t \leq T.$$

The equilibrium point is then explicitly given by

$$\hat{u}(t) = \frac{\alpha - r}{\gamma\sigma^2}\left(e^{r(T-t)} + \frac{(\alpha - r)^2}{r\gamma\sigma^2}\left(e^{r(T-t)} - 1\right)\right)^{-1} x, \quad 0 \le t \le T.$$

Variational equations

Let $y^\epsilon(\cdot)$ and $z^\epsilon(\cdot)$ be respectively the solutions of the following SDEs:

$$\begin{cases} dy^\epsilon(t) = \{b_x(t)y^\epsilon(t) + b_y(t)E\left[y^\epsilon(t)\right] + \delta b(t)\mathbb{1}_{E_\epsilon}(t)\}\,dt \\ \qquad + \{\sigma_x(t)y^\epsilon(t) + \sigma_y(t)E\left[y^\epsilon(t)\right] + \delta\sigma(t)\mathbb{1}_{E_\epsilon}(t)\}\,dB(t), \\ y^\epsilon(0) = 0, \end{cases} \tag{53}$$

$$\begin{cases} dz^\epsilon(t) = \{b_x(t)z^\epsilon(t) + b_y(t)E\left[z^\epsilon(t)\right] + \mathcal{L}_t(b, y^\epsilon) + \delta b_x(t)y^\epsilon(t)\mathbb{1}_{E_\epsilon}(t)\}\,dt \\ \qquad + \{\sigma_x(t)z^\epsilon(t) + \sigma_y(t)E\left[z^\epsilon(t)\right] + \mathcal{L}_t(b, y^\epsilon) + \delta b_x(t)y^\epsilon(t)\mathbb{1}_{E_\epsilon}(t)\}\,dB(t), \\ z^\epsilon(0) = 0. \end{cases} \tag{54}$$

$$\mathcal{L}_t(\varphi, y) = \frac{1}{2}\varphi_{xx}(t, \hat{x}(t), E[\hat{x}(t)], \hat{u}(t))y^2.$$

Duality

Lemma. We have

$$E\left[p(T)y^{\epsilon}(T)\right] = E\left[\int_0^T y^{\epsilon}(t)\left(h_x(t)+E[h_y(t)]\right)dt\right] + E\left[\int_0^T \left(p(t)\delta b(t)+q(t)\delta\sigma(t)\right)I_{E_\epsilon}(t)dt\right], \quad (55)$$

and

$$E\left[p(T)z^{\epsilon}(T)\right] = E\left[\int_0^T z^{\epsilon}(t)(h_x(t)+E[h_y(t)])dt\right] + E\left[\int_0^T \left(p(t)\delta b_x(t)+q(t)\delta\sigma_x(t)\right)y^{\epsilon}(t)I_{E_\epsilon}(t)dt\right] + E\left[\int_0^T \left(p(t)\mathcal{L}(b,y^{\epsilon}(t))+q(t)\mathcal{L}_t(\sigma,y^{\epsilon})\right)dt\right]. \quad (56)$$

Taylor expansions and estimates

Let

$$dx^{\epsilon}(t) = b(t, x^{\epsilon}(t), E\left[x^{\epsilon}(t)\right], u_t^{\epsilon})dt + \sigma(t, x^{\epsilon}(t), E\left[x^{\epsilon}(t)\right], u_t^{\epsilon})dB(t),$$

$$d\hat{x}(t) = b(t, \hat{x}(t), E\left[\hat{x}(t)\right], \hat{u}_t)dt + \sigma(t, \hat{x}(t), E\left[\hat{x}(t)\right], \hat{u}_t)dB(t),$$

whith $x^{\epsilon}(0) = \hat{x}(0) = x_0$.

Proposition. For any $k \geq 1$,

$$E\left[\sup_{t\in[0,T]} \left|x^{\epsilon}(t) - \hat{x}(t)\right|^{2k}\right] \leq C_k \epsilon^k, \tag{57}$$

$$E\left[\sup_{t\in[0,T]} \left|y^{\epsilon}(t)\right|^{2k}\right] \leq C_k \epsilon^k, \tag{58}$$

$$E\left[\sup_{t\in[0,T]}|z^{\epsilon}(t)|^{2k}\right]\le C_k\epsilon^{2k}, \tag{59}$$

$$E\left[\sup_{t\in[0,T]}|x^{\epsilon}(t)-(\hat{x}(t)+y^{\epsilon}(t))|^{2k}\right]\le C_k\epsilon^{2k}, \tag{60}$$

$$E\left[\sup_{t\in[0,T]}|x^{\epsilon}(t)-(\hat{x}(t)+y^{\epsilon}(t)+z^{\epsilon}(t))|^{2k}\right]\le C_k\epsilon^{2k}\rho_k(\epsilon), \tag{61}$$

where, $\rho_k:(0,\infty)\to(0,\infty)$ is such that $\rho_k(\epsilon)\downarrow 0$ as $\epsilon\downarrow 0$.

$$\sup_{t\in[0,T]}|E[y^{\epsilon}(t)]|^2\le\epsilon\rho(\epsilon),\ \ \epsilon>0, \tag{62}$$

for some function $\rho:(0,\infty)\to(0,\infty)$ such that $\rho(\epsilon)\downarrow 0$ as $\epsilon\downarrow 0$.

Estimate (62) is derived as a consequence of the following result.

Lemma. For any progressively measurable process $(\Phi(t))_{t\in[0,T]}$ for which, for all $p \geq 1$, there exists a positive constant C_p, such that

$$E[\sup_{t\in[0,T]} |\Phi(t)|^p] \leq C_p, \tag{63}$$

there exists a function $\tilde{\rho} : (0,\infty) \to (0,\infty)$ with $\tilde{\rho}(\epsilon) \to 0$ as $\epsilon \downarrow 0$, such that

$$|E[\Phi(T)y^{\epsilon}(T)]|^2 + \int_0^T |E[\Phi(s)y^{\epsilon}(s)]|^2\, ds \leq \epsilon\tilde{\rho}(\epsilon), \quad \epsilon > 0. \tag{64}$$